

PERRON'S METHOD FOR PATHWISE VISCOSITY SOLUTIONS

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ABSTRACT. We use Perron's method to construct solutions of fully nonlinear degenerate parabolic pathwise partial differential equations. The proof uses an adaptation of the argument from the classical viscosity setting, and depends on a finite speed of propagation property for smooth solutions of pathwise Hamilton-Jacobi equations.

1. INTRODUCTION

For $T > 0$, we construct solutions for the nonlinear parabolic pathwise partial differential equation

$$(1) \quad du = F(D^2u, Du, u, x, t) dt + \sum_{i=1}^m H^i(Du, x) \cdot dW^i \quad \text{in } \mathbb{R}^n \times (0, T], \quad u(\cdot, 0) = u_0 \quad \text{on } \mathbb{R}^n.$$

The initial condition u_0 lies in $BUC(\mathbb{R}^n)$, the space of bounded, uniformly continuous functions on \mathbb{R}^n , and $W = (W^1, W^2, \dots, W^m) : [0, \infty) \rightarrow \mathbb{R}^m$ is a continuous path.

When W is continuously differentiable, or even of bounded variation, dW stands for $\frac{d}{dt}W(t) = \dot{W}_t$, the time-derivative of W , which is respectively continuous or in $L^1([0, T])$, and the symbol \cdot is just standard multiplication. In either case, the viscosity theory for (1) is well-established, as documented by Crandall, Ishii, and Lions [1] in the former case, or by Ishii [2] and Lions and Perthame [3] in the latter.

In this paper, we allow W to be merely continuous, the example of most interest being a sample path of a stochastic process, for instance Brownian motion. In this case, W is nowhere differentiable, and, in fact, has unbounded variation on every interval. For such paths, the symbol \cdot is regarded as the Stratonovich differential. More generally, W may be a geometric rough path, a special case being Brownian motion enhanced with its Stratonovich iterated integrals. At the very least, we require W to be such that certain ordinary differentiable equations driven by W have a pathwise solution theory. We give more details about this point when we list the assumptions.

The notion of pathwise solutions for equations like (1) was developed by Lions and Souganidis in the papers [4], [5], and [6], and in the forthcoming book [8]. In their work, existence is established for (1) by extending the solution operator, which is well-defined when W is C^1 , to continuous paths. More precisely, it is shown that, for a C^1 -family $\{W^\eta\}_{\eta>0}$ that satisfies $\lim_{\eta \rightarrow 0} W^\eta = W$ uniformly in $[0, T]$, if u^η is the classical viscosity solution of

$$(2) \quad u_t^\eta = F(D^2u^\eta, Du^\eta, u^\eta, x, t) + \sum_{i=1}^m H^i(Du^\eta, x) \dot{W}_t^{i,\eta} \quad \text{in } \mathbb{R}^n \times (0, T], \quad u^\eta(\cdot, 0) = u_0 \quad \text{on } \mathbb{R}^n,$$

then the sequence $\{u^\eta\}_{\eta>0}$ is Cauchy, and, in fact, converges locally uniformly to a unique limit u , independently of the approximating family $\{W^\eta\}_{\eta>0}$. The function u can then be shown to be the unique pathwise viscosity solution of (1).

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Since pathwise viscosity solutions are defined in terms of sub- and super-solutions, it is often possible to prove a comparison principle. That is, for a sub- and super-solution u and v , we have, for all $t > 0$, $\sup_{\mathbb{R}^n} (u(\cdot, t) - v(\cdot, t))_+ \leq \sup_{\mathbb{R}^n} (u(\cdot, 0) - v(\cdot, 0))_+$. A consequence is the following principle of Perron:

Theorem 1. *The unique solution of (1) is given by the maximal sub-solution (or minimal super-solution).*

The proof is immediate: in view of the comparison principle, the solution of (1) is greater than or equal to any sub-solution, and is itself a sub-solution.

The purpose of this paper is to give a direct proof of Theorem 1. In other words, we give a constructive argument without appealing to the existence of solutions, and thus give an alternative method for proving existence for (1), in situations for which the comparison principle is already established. This may be an advantage when it is difficult to obtain estimates for (2) that allow one to pass to the limit as $\eta \rightarrow 0$.

The paper is organized as follows. The main assumptions on F , H , and W are given in Section 2, along with the definition of pathwise viscosity solutions. In Section 3, we divide the proof of the Perron construction into two main steps, given as Propositions 1 and 2. The proofs of Propositions 1 and 2 appear in respectively Sections 4 and 5, while the construction of specific sub- and super-solutions is described in Section 6. Finally, in the appendix, we give examples of Hamiltonians H and paths W for which solutions of (1) satisfy the comparison principle.

Notation. $C_b^2(\mathbb{R}^n)$ is the space of functions with bounded and continuous first and second derivatives. $USC(\mathbb{R}^n \times [0, T])$ and $LSC(\mathbb{R}^n \times [0, T])$ are the spaces of respectively upper and lower semicontinuous functions on $\mathbb{R}^n \times [0, T]$.

For $U : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$, the upper-semicontinuous and lower-semicontinuous envelopes U^* and U_* of U are defined as respectively

$$U^*(x, t) := \limsup_{(y, s) \rightarrow (x, t)} U(y, s) \quad \text{and} \quad U_*(x, t) := \liminf_{(y, s) \rightarrow (x, t)} U(y, s).$$

S_n is the space of symmetric n -by- n matrices, and if $X, Y \in S_n$, the inequality $X \leq Y$ means that $X\xi \cdot \xi \leq Y\xi \cdot \xi$ for all $\xi \in \mathbb{R}^n$. For $x \in \mathbb{R}$, $x_+ := \max(x, 0)$.

If $K \subset \mathbb{R}^n$ is a closed set and $r > 0$, the closed set K_r is defined by $K_r := \{x \in \mathbb{R}^n \mid \text{dist}(x, K^c) \geq r\}$. For $r, s, t_0 > 0$ and $x_0 \in \mathbb{R}^n$, we set $B_r(x_0) := \{x \in \mathbb{R}^n \mid |x - x_0| < r\}$, $N_{r,s}(x_0, t_0) := B_r(x_0) \times (t_0 - s, t_0 + s)$, and $N_r(x_0, t_0) := N_{r,r}(x_0, t_0)$.

2. ASSUMPTIONS

We assume that F , a function of $(X, p, r, x, t) \in S_n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times [0, \infty)$, is continuous, bounded for bounded (X, p, r) , degenerate elliptic, and nonincreasing in r ; that is,

$$(3) \quad \begin{cases} F \in C(S_n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times [0, \infty)) \cap L^\infty(B_R \times B_R \times B_R \times \mathbb{R}^n \times [0, T]) & \text{for all } R, T > 0, \\ X \mapsto F(X, \cdot, \cdot, \cdot, \cdot) \text{ is nondecreasing, and } r \mapsto F(\cdot, \cdot, r, \cdot, \cdot) \text{ is nonincreasing.} \end{cases}$$

The Hamiltonians H^i require stronger regularity than F . This has to do with the system of characteristic equations. For $\phi \in C_b^2(\mathbb{R}^n)$ and $t_0 \in [0, T]$, these are given by

$$(4) \quad \begin{cases} dX = -\sum_{i=1}^m D_p H^i(P, X) \cdot dW^i & X(x, t_0) = x \\ dP = \sum_{i=1}^m D_x H^i(P, X) \cdot dW^i & P(x, t_0) = D\phi(x) \\ dZ = \sum_{i=1}^m (H^i(P, X) - D_p H^i(P, X) \cdot P) \cdot dW^i & Z(x, t_0) = \phi(x). \end{cases}$$

In what follows, it will be necessary for (4) to have a pathwise theory that yields smooth flows. More precisely, we assume that H^i is such that

$$(5) \quad \text{there exists a unique solution } (X, P, Z) \in C((-\infty, \infty); C_b^2(\mathbb{R}^n; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})) \text{ of (4).}$$

We also impose that the differential operator d satisfies the chain rule. In other words, for any smooth f and for $B \in C([0, \infty))$ equal to X , P , Z , or W ,

$$(6) \quad df(B) = Df(B) \cdot dB.$$

In view of the regularity of the flow given by (5), for each $\phi \in C_b^2(\mathbb{R}^n)$, there exists $h > 0$ depending only on $\|\phi\|_{C_b^2(\mathbb{R}^n)}$ such that $x \mapsto X(x, t)$ is invertible whenever $t \in (t_0 - h, t_0 + h)$, and the inverse $x \mapsto X^{-1}(x, t)$ is also C^2 . It is then standard, using the chain rule property (6), that defining $\Phi \in C((t_0 - h, t_0 + h); C_b^2(\mathbb{R}^n))$ implicitly by $\Phi(X(x, t), t) := Z(x, t)$ leads to a solution of the pathwise Hamilton-Jacobi equation

$$(7) \quad d\Phi = \sum_{i=1}^m H^i(D\Phi, x) \cdot dW^i \quad \text{in } \mathbb{R}^n \times (t_0 - h, t_0 + h), \quad \Phi(\cdot, t_0) = \phi \quad \text{on } \mathbb{R}^n.$$

If $W \in C^1([0, T]; \mathbb{R}^m)$, then (4) is a system of time-inhomogeneous ordinary differential equations, and (6) trivially holds. A more interesting example is when W is a Brownian motion. In this case, (5) and (6) will hold if (4) is interpreted with the Stratonovich differential. In general, W may be a geometric rough path, in which case it is not hard to determine the regularity on H for which the above properties hold.

Finally, when there is a single path and Hamiltonian (that is, $m = 1$), the solution (X, P, Z) is given by $(X, P, Z)(x, t) = (X_d, P_d, Z_d)(x, W_t - W_{t_0})$, where (X_d, P_d, Z_d) solve the time-homogenous ordinary differential equations

$$(8) \quad \begin{cases} \dot{X}_d = -D_p H(P_d, X_d) & X_d(x, 0) = x \\ \dot{P}_d = D_x H(P_d, X_d) & P_d(x, 0) = D\phi(x) \\ \dot{Z}_d = H(P_d, X_d) - D_p H(P_d, X_d) \cdot P_d & Z_d(x, 0) = \phi(x). \end{cases}$$

Indeed, the solution Φ of (7) is given by $\Phi(x, t) = U(x, W_t - W_{t_0})$, where U is a smooth (locally in time) solution of the time-independent Hamilton Jacobi equation $U_t = H(DU, x)$ with $U(x, 0) = \phi$. This is certainly true for $W \in C^1$, and extends to the case of geometric rough paths by density, or by using the chain rule property for such paths. For a general continuous path W , this formula can be seen as the definition of a smooth-in- x solution of (7).

We will use the notation $S(t, t_0)\phi(x) := \Phi(x, t)$ for solutions of (7), so that $S(t, t_0)$ maps $C_b^2(\mathbb{R}^n)$ to $C_b^2(\mathbb{R}^n)$ whenever $t \in (t_0 - h, t_0 + h)$. The comparison principle holds for this solution operator, that is, for any $t \in (t_0 - h, t_0 + h)$ and $\phi_1, \phi_2 \in C_b^2(\mathbb{R}^n)$, $\sup_{\mathbb{R}^n} (S(t, t_0)\phi_1 - S(t, t_0)\phi_2) \leq \sup_{\mathbb{R}^n} (\phi_1 - \phi_2)$. This result is classical for C^1 paths W . In all of the cases discussed above, the most general of which is the geometric rough path, the solution of (4) can be obtained as the uniform limit of solutions to (4) where W is replaced by a particular C^1 -approximation to W . It follows that the comparison principle extends to this case as well.

The last assumption for (4) is that

$$(9) \quad \begin{cases} \text{for every } R > 0, \text{ there exists a nondecreasing, continuous } \rho_R : [0, \infty) \rightarrow [0, \infty) \text{ such that,} \\ \text{for all } \phi \in C_b^2(\mathbb{R}^n) \text{ with } \|\phi\|_{C_b^2(\mathbb{R}^n)} \leq R, t \in (t_0 - h, t_0 + h), \text{ and } x \in \mathbb{R}^n, \\ |X^{-1}(x, t) - x| \leq \rho_R(|t - t_0|). \end{cases}$$

When $m = 1$, it is clear that we can take $\rho_R(s) := \sup_{|p| \leq R, x \in \mathbb{R}^n} |D_p H(p, x)| s$. In general, one may use estimates from the theory of rough differential equations to track the deviation of the X characteristic.

Using (9), we prove the following domain of dependence result for the solution operator $S(t, t_0)$:

Lemma 1. For $\phi_1, \phi_2 \in C_b^2(\mathbb{R}^n)$, set $R := \max \left(\|\phi_1\|_{C_b^2(\mathbb{R}^n)}, \|\phi_2\|_{C_b^2(\mathbb{R}^n)} \right)$. Let $t_0 \in [0, T]$ and let $h > 0$ be such that (7) is solvable for ϕ_1 and ϕ_2 . Fix a closed set $K \subset \mathbb{R}^n$. Then, for all $t \in (t_0 - h, t_0 + h)$ satisfying $\rho_R(|t - t_0|) < \max_{x \in K^c} \text{dist}(x, K^c)$,

$$\sup_{K_{\rho_R(|t-t_0|)}} (S(t, t_0)\phi_1 - S(t, t_0)\phi_2) \leq \sup_K (\phi_1 - \phi_2).$$

Proof. We first prove that

$$(10) \quad \text{if } \phi_1 = \phi_2 \text{ in } K, \quad \text{then } S(t, t_0)\phi_1 = S(t, t_0)\phi_2 \text{ in } K_{\rho_R(|t-t_0|)}.$$

For $i = 1, 2$, let (X_i, P_i, Z_i) be the solution of (4) for ϕ_i . Let $x \in K_{\rho_R(|t-t_0|)}$. In view of (9), $y := X_1^{-1}(x, t)$ lies in the interior of K . This implies that $\phi_1(y) = \phi_2(y)$ and $D\phi_1(y) = D\phi_2(y)$, so $(X_1, P_1, Z_1)(y, t) = (X_2, P_2, Z_2)(y, t)$ and thus $y = X_2^{-1}(x, t)$. We conclude that

$$S(t, t_0)\phi_1(x) = Z_1(X_1^{-1}(x, t)) = Z_1(y) = Z_2(y) = Z_2(X_2^{-1}(x, t)) = S(t, t_0)\phi_2(x).$$

Now assume $\phi_1 \leq \phi_2$ in K . Fix $\epsilon > 0$. Let $\tilde{\phi}_2 \in C_b^2(\mathbb{R}^n)$ be such that $\phi_2 = \tilde{\phi}_2$ in K and $\phi_1 \leq \tilde{\phi}_2 + \epsilon$ in \mathbb{R}^n . Then the comparison principle for $S(t, t_0)$ and (10) immediately give, for all $x \in K_{\rho_R(|t-t_0|)}$,

$$S(t, t_0)\phi_1(x) \leq S(t, t_0)(\tilde{\phi}_2 + \epsilon)(x) = S(t, t_0)(\phi_2 + \epsilon)(x) = S(t, t_0)\phi_2(x) + \epsilon.$$

Letting $\epsilon \rightarrow 0$ gives the result.

Finally, for the general case, we note that $\phi_1 \leq \phi_2 + \sup_K(\phi_1 - \phi_2)$ in K and use once more the fact that $S(t, t_0)$ commutes with constants. \square

The solution operator $S(t, t_0)$ is used to define sub- and super-solutions of the original problem (1). In particular, in analogy with the classical viscosity theory, test functions of the form $S(t, t_0)\phi$ are compared to the sub- and super-solutions, with the effect that the “rough part” of (1), that is, the term involving dW^i , is cancelled out, allowing for the use of classical viscosity inequalities.

Definition 1. We say $u \in USC(\mathbb{R}^n \times [0, T])$ (resp. $u \in LSC(\mathbb{R}^n \times [0, T])$) is a pathwise viscosity sub-solution (resp. super-solution) of (1) if $u \leq u_0$ (resp. $u \geq u_0$) on $\mathbb{R}^n \times [0, T]$, and, whenever $\phi \in C_b^2(\mathbb{R}^n)$, $\psi \in C^1(0, \infty)$, and $u(x, t) - S(t, t_0)\phi(x) - \psi(t)$ attains a local maximum (resp. minimum) at $(x_0, t_0) \in \mathbb{R}^n \times (0, T]$, then

$$\begin{aligned} \psi'(t_0) &\leq F(D^2\phi(x_0, t_0), D\phi(x_0, t_0), u(x_0, t_0), x_0, t_0) \\ (\text{resp. } \psi'(t_0) &\geq F(D^2\phi(x_0, t_0), D\phi(x_0, t_0), u(x_0, t_0), x_0, t_0)). \end{aligned}$$

A solution of (1) is both a sub- and super-solution.

As in the deterministic case, in the definition above, it suffices to replace local maxima or minima with strict maxima or minima. This tool is useful in stability arguments. It also suffices to replace maxima or minima over neighborhoods $B_r(x_0) \times (t_0 - h, t_0 + h)$ with maxima or minima over semi-closed neighborhoods like $B_r(x_0) \times (t_0 - h, t_0]$, by subtracting penalizations of the form $\nu(t_0 - t)^{-1}$ as $\nu \rightarrow 0$.

We remark that, for $\phi \in C_b^2(\mathbb{R}^n)$, $S(t, t_0)\phi$ is a pathwise viscosity solution of (7). This has been shown in [8] when W is a Brownian motion, using the definition of Stratonovich integrals via rough path integration. The argument easily extends to the geometric rough path case. It is also true that, if $W \in C^1$, then u is a solution of (1) in the sense of Definition 1 if and only if it is a classical viscosity solution.

The main assumption we make for (1) is that the comparison principle holds, that is,

$$(11) \quad \begin{cases} \text{if } u \text{ and } v \text{ are respectively a sub- and super-solution of (1), then} \\ \sup_{x \in \mathbb{R}^n} (u(x, t) - v(x, t))_+ \leq \sup_{x \in \mathbb{R}^n} (u(x, 0) - v(x, 0))_+. \end{cases}$$

3. MAIN STEPS FOR PERRON'S METHOD

We summarize the two steps of Perron's method in the following propositions, much in the same way as in Section 4 of [1].

The first step is to prove the stability of sub-solutions and super-solutions under respectively suprema and infima. It is clear from the definition that the maximum (resp. minimum) of a finite number of sub-solutions (resp. super-solutions) is also a sub-solution (resp. super-solution). Proposition 1 is the generalization of this observation to infinite families.

Proposition 1. *Let \mathcal{F} be a family of sub- (resp. super-) solutions of (1). Define*

$$U(x, t) := \sup_{v \in \mathcal{F}} \left(\text{resp. } \inf_{v \in \mathcal{F}} \right) v(x, t).$$

Assume that $U^ < \infty$ (resp. $U_* > -\infty$). Then U^* (resp. U_*) is a sub- (resp. super-) solution of (1).*

The next step consists of verifying that the maximal sub-solution, constructed using Proposition 1, is in fact a super-solution.

Proposition 2. *Suppose that w is a sub-solution of (1), and that w_* fails to be a super-solution at some $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$. Then, for each $\kappa > 0$, there exists a sub-solution w_κ of (1) satisfying*

$$\begin{cases} w_\kappa \geq w, \sup(w_\kappa - w) > 0, \text{ and} \\ w_\kappa = w \text{ for } (x, t) \notin N_\kappa(x_0, t_0). \end{cases}$$

The following Lemma, whose proof we discuss in Section 6, asserts the existence of sub- and super-solutions.

Lemma 2. *For every $u_0 \in BUC(\mathbb{R}^d)$, there exist a locally bounded sub- and super-solution \underline{u} and \overline{u} of (1) satisfying $\underline{u}_*(\cdot, 0) = \overline{u}^*(\cdot, 0) = u_0$.*

We now define

$$(12) \quad u(x, t) := \sup \{v(x, t) : v \text{ is a sub-solution of (1)}\}.$$

The following Theorem is proved using the preceding Propositions and Lemmas.

Theorem 2. *The function u given by (12) is the unique pathwise viscosity solution of (1).*

Proof. Since $u_0 = \underline{u}_*(\cdot, 0) \leq u_*(\cdot, 0) \leq u(\cdot, 0) \leq u^*(\cdot, 0) \leq \overline{u}^*(\cdot, 0) = u_0$, we have $u(\cdot, 0) = u_0$, so the initial condition is satisfied.

In view of the comparison principle, $u^* \leq \overline{u}^* < \infty$ because \overline{u} is locally bounded. Proposition 1 then yields that u^* is a sub-solution of (1). By definition, $u^* \leq u$, and therefore $u^* = u$; that is, u is itself upper semicontinuous and a sub-solution.

We also have $u_* \geq \underline{u}_* > -\infty$. Assume that u_* is not a super-solution. Then, by Proposition 2, there exists a sub-solution $\tilde{u} \geq u$ and a neighborhood $N \subset \mathbb{R}^n \times (0, T]$ such that $\tilde{u} = u$ in $(\mathbb{R}^n \times [0, T]) \setminus N$ and $\sup_N(\tilde{u} - u) > 0$, contradicting the maximality of u . Therefore, u_* is a super-solution of (1).

The comparison principle gives $u^* \leq u_*$. By definition, we have $u_* \leq u^*$, and therefore $u = u_* = u^*$ is a solution with $u = u_0$ on $\mathbb{R}^n \times \{0\}$. The uniqueness of u follows from another application of the comparison principle. \square

4. PROOF OF PROPOSITION 1

We prove Proposition 1 only for sub-solutions, since the corresponding proof for super-solutions is almost identical.

Let $\phi \in C_b^2(\mathbb{R}^n)$ and $\psi \in C^1((0, \infty))$, and let $t_0 > 0$ and $h > 0$ be such that $S(t, t_0)\phi$ is defined and in $C_b^2(\mathbb{R}^n)$ for all $t \in (t_0 - h, t_0 + h)$. Assume that $U^*(x, t) - S(t, t_0)\phi(x) - \psi(t)$ attains a local maximum at $(x_0, t_0) \in \mathbb{R}^N \times (t_0 - h, t_0 + h)$. Without loss of generality, we may take $x_0 = 0$, $\phi(0) = 0$, and $\psi(t_0) = 0$. Set $p = D\phi(0)$, $X = D^2\phi(0)$, and $a = \psi'(t_0)$.

For fixed $\delta > 0$, let $r > 0$ be such that, for all $(x, t) \in N_{2r}(0, t_0)$,

$$\phi(x) \leq \langle p, x \rangle + \frac{1}{2} \langle Xx, x \rangle + \delta|x|^2 \quad \text{and} \quad \psi(t) \leq a(t - t_0) + \delta|t - t_0|.$$

Because ϕ is bounded, we can find ϕ_1 and ϕ_2 in C_b^2 such that

$$\begin{cases} \phi_1(x) = \langle p, x \rangle + \frac{1}{2} \langle Xx, x \rangle + \delta|x|^2 & \text{on } B_{2r}(x_0), \\ \phi_2(x) = \langle p, x \rangle + \frac{1}{2} \langle Xx, x \rangle + 2\delta|x|^2 & \text{on } B_{2r}(x_0), \text{ and} \\ \phi \leq \phi_1 \leq \phi_2 & \text{on } \mathbb{R}^n. \end{cases}$$

The third inequality implies that, for all $t \in (t_0 - h, t_0 + h)$, $S(t, t_0)\phi \leq S(t, t_0)\phi_1 \leq S(t, t_0)\phi_2$. In particular, because $\phi(0) = \phi_1(0)$, the function $U^*(x, t) - S(t, t_0)\phi_1(x) - \psi(t)$ still attains a local maximum at $(0, t_0)$.

By definition, there exist sequences $(x_n, t_n) \in \mathbb{R}^n \times (t_0 - h, t_0 + h)$ and $v_n \in \mathcal{F}$ such that, as $n \rightarrow \infty$, $(x_n, t_n) \rightarrow (0, t_0)$ and $v_n(x_n, t_n) \rightarrow U^*(0, t_0)$. The upper semicontinuous function

$$(13) \quad v_n(x, t) - S(t, t_0)\phi_2(x) - a(t - t_0) - 2\delta(|t - t_0|^2 + n^{-1})^{1/2}$$

attains its maximum at some $(x'_n, t'_n) \in \overline{N_r(0, t_0)}$. In particular,

$$(14) \quad \begin{aligned} v_n(x_n, t_n) &\leq v_n(x'_n, t'_n) + S(t_n, t_0)\phi_2(x_n) - S(t'_n, t_0)\phi_2(x'_n) + a(t_n - t'_n) \\ &\quad + 2\delta \left((|t_n - t_0|^2 + n^{-1})^{1/2} - (|t'_n - t_0|^2 + n^{-1})^{1/2} \right). \end{aligned}$$

A subsequence of (x'_n, t'_n) converges to some $(y, s) \in \overline{N_r(0, t_0)}$, and, passing to the limit in (14), we find

$$(15) \quad \begin{aligned} U^*(0, t_0) &\leq U^*(y, s) - S(s, t_0)\phi_2(y) - a(s - t_0) - 2\delta|s - t_0| \\ &\leq U^*(0, t_0) + S(s, t_0)\phi_1(y) - S(s, t_0)\phi_2(y) - \delta|s - t_0| \leq U^*(0, t_0) - \delta|s - t_0|. \end{aligned}$$

It follows that $\delta|s - t_0| \leq 0$, so that $s = t_0$. Inserting this fact into (15), we obtain $\phi_2(y) \leq \phi_1(y)$, and, because $y \in B_r(x_0)$,

$$\langle p, y \rangle + \frac{1}{2} \langle Xy, y \rangle + 2\delta|y|^2 \leq \langle p, y \rangle + \frac{1}{2} \langle Xy, y \rangle + \delta|y|^2,$$

which implies $y = 0$. It follows that the full sequence (x'_n, t'_n) converges, as $n \rightarrow \infty$, to $(0, t_0)$, and $\lim_{n \rightarrow \infty} v_n(x'_n, t'_n) = U^*(0, t_0)$. Therefore, for sufficiently large n , $(x'_n, t'_n) \in N_r(0, t_0)$, so that the maximum attained in (13) is an interior maximum.

Set $\Phi(x, t) := S(t, t_0)\phi_2(x)$. Definition 1 and (13) yield

$$a + 2\delta \frac{t'_n - t_0}{(|t'_n - t_0|^2 + n^{-1})^{-1}} \leq F(D^2\Phi(x'_n, t'_n), D\Phi(x'_n, t'_n), v_n(x'_n, t'_n), x'_n, t'_n).$$

Letting $n \rightarrow \infty$ and $\delta \rightarrow 0$ yields $a \leq F(X, p, u(0, t_0), 0, t_0)$ as desired.

5. PROOF OF PROPOSITION 2

Assume without loss of generality that $x_0 = 0$. Since w_* is not a super-solution at $(0, t_0)$, there exist $\phi \in C_b^2(\mathbb{R}^n)$ and $\psi \in C^1((0, \infty))$ such that $w_*(x, t) - S(t, t_0)\phi(x) - \psi(t)$ attains a local minimum at $(0, t_0)$, but

$$(16) \quad \psi'(t_0) - F(D^2\phi(0), D\phi(0), w_*(0, t_0), 0, t_0) < 0.$$

Assume again $\phi(0) = 0$ and $\psi(t_0) = 0$, and set $X := D^2\phi(0)$, $p := D\phi(0)$, and $a := \psi'(t_0)$. We also define $\omega_1, \omega_2 : [0, \infty) \rightarrow [0, \infty)$ by

$$\omega_1(s) := \sup_{|x| \leq s} \frac{|\phi(x) - \langle p, x \rangle - \frac{1}{2}\langle Xx, x \rangle|}{|x|^2} \quad \text{and} \quad \omega_2(s) := \sup_{|t-t_0| \leq s} \frac{|\psi(t) - a(t-t_0)|}{|t-t_0|},$$

so that ω_i is nondecreasing and $\lim_{s \rightarrow 0+} \omega_i(s) = 0$.

Let $0 < \gamma < 1$, let $0 < r, s < \kappa$ be such that

$$(17) \quad \omega_1(r), \omega_2(s) \leq \frac{\gamma}{2},$$

and set

$$(18) \quad \delta := \gamma \min \left(\frac{r^2}{16}, \frac{s}{8} \right).$$

Let $\eta_{\gamma,r} \in C_b^2(\mathbb{R}^n)$ be such that

$$\eta_{\gamma,r}(x) = \langle p, x \rangle + \frac{1}{2}\langle Xx, x \rangle - \gamma|x|^2 \text{ in } B_r(x_0) \quad \text{and} \quad \eta_{\gamma,r} \leq \phi \text{ in } \mathbb{R}^n,$$

and set

$$\hat{w}(x, t) := w_*(0, t_0) + \delta + S(t, t_0)\eta_{\gamma,r}(x) + a(t - t_0) - \gamma(|t - t_0|^2 + \delta^2)^{1/2},$$

where we suppress the dependence of \hat{w} on the various parameters. In view of the strict inequality in (16), the continuity of the solution map $S(t, t_0)$ on C_b^2 , and the continuity of F , it follows that \hat{w} is a sub-solution of (1) in $N_{r,s}(0, t_0)$, shrinking γ , r , and s (and therefore δ) if necessary.

Observe that there exists a sequence $(x_n, t_n) \rightarrow (0, t_0)$ such that $w(x_n, t_n) \rightarrow w_*(0, t_0)$, and so

$$\lim_{n \rightarrow \infty} (w(x_n, t_n) - \hat{w}(x_n, t_n)) = -(1 - \gamma)\delta < 0.$$

Therefore, there exist points in $N_{r,s}(0, t_0)$ arbitrarily close to $(0, t_0)$ for which $w < \hat{w}$.

Finally, define

$$R := \sup_{|t-t_0| \leq 1} \max \left\{ \|S(t, t_0)\phi\|_{C_b^2(\mathbb{R}^n)}, \|S(t, t_0)\eta_{\gamma,r}\|_{C_b^2(\mathbb{R}^n)} \right\},$$

and shrink s (and therefore δ) further so that

$$(19) \quad \rho_R(s) \leq \frac{r}{8}.$$

We claim that

$$(20) \quad w(x, t) > \hat{w}(x, t) \quad \text{in } N_{7r/8,s}(0, t_0) \setminus N_{5r/8,s/2}(0, t_0).$$

Observe that

$$w(x, t) - \hat{w}(x, t) \geq w_*(x, t) - \hat{w}(x, t) \geq -\delta + S(t, t_0)\phi(x) - S(t, t_0)\eta_{\gamma,r}(x) + (\gamma - \omega_2(s))|t - t_0|.$$

First suppose that $s/2 < |t - t_0| < s$ and $|x| < 7r/8$. Then (17) and (18) give

$$w(x, t) - \hat{w}(x, t) \geq -\delta + (\gamma - \omega_2(s)) \cdot \frac{s}{2} \geq -\delta + \frac{\gamma s}{4} \geq \frac{\gamma s}{8} > 0.$$

Meanwhile, (19) and Lemma 1 applied to the annulus $K = \overline{B_r(0)} \setminus B_{r/2}(0)$ yield

$$\begin{aligned} \inf_{r/2 \leq |x| \leq r} (\phi(x) - \eta_{\gamma,r}(x)) &\leq \inf \{S(t, t_0)\phi(x) - S(t, t_0)\eta_{\gamma,r}(x) : \text{dist}(x, K^c) \geq \rho_R(|t - t_0|)\} \\ &\leq \inf \left\{ S(t, t_0)\phi(x) - S(t, t_0)\eta_{\gamma,r}(x) : \min \left(r - |x|, |x| - \frac{r}{2} \right) \geq \rho_R(s) \right\} \\ &\leq \inf \{S(t, t_0)\phi(x) - S(t, t_0)\eta_{\gamma,r}(x) : 5r/8 < |x| < 7r/8\}. \end{aligned}$$

Combining this with (17) and (18), we see that, whenever $5r/8 < |x| < 7r/8$ and $|t - t_0| < s$,

$$w(x, t) - \hat{w}(x, t) \geq -\delta + \inf_{r/2 \leq |x| \leq r} (\phi(x) - \eta_{r,\gamma}(x)) \geq -\delta + (\gamma - \omega_1(r)) \cdot \left(\frac{r}{2}\right)^2 \geq -\delta + \frac{\gamma r^2}{8} \geq \frac{\gamma r^2}{16} > 0.$$

This finishes the proof of (20).

Finally, we define

$$w_\kappa(x, t) := \begin{cases} \max(\hat{w}(x, t), w(x, t)) & (x, t) \in N_{7r/8,s}(0, t_0) \\ w(x, t) & (x, t) \notin N_{7r/8,s}(0, t_0). \end{cases}$$

Clearly $w_\kappa = w$ outside of $N_\kappa(0, t_0)$. We have already seen that there exist points such that $w_\kappa(x, t) > w(x, t)$. Finally, in view of (20), $w_\kappa = w$ in a neighborhood of the boundary of $N_{7r/8,s}(0, t_0)$. These facts imply that w_κ is the desired sub-solution.

6. PROOF OF LEMMA 2: EXISTENCE OF SUB- AND SUPER-SOLUTIONS

In certain situations, constructing the solutions \underline{u} and \overline{u} is very simple. For example, suppose $F \equiv 0$, H is x -independent, and $u_0 \in C_b^2(\mathbb{R}^n)$. Let Φ and $h > 0$ be such that Φ is the short-time, smooth-in- x solution of (7) for $0 \leq t < h$. Then we may take

$$\overline{u}(x, t) = \begin{cases} \min(\Phi(x, t) + Ct, \|u_0\|_\infty + \sum_{i=1}^m H^i(0)W_t^i), & 0 \leq t < h \\ \|u_0\|_\infty + \sum_{i=1}^m H^i(0)W_t^i, & t \geq h \end{cases}$$

and

$$\underline{u}(x, t) = \begin{cases} \max(\Phi(x, t) - Ct, -\|u_0\|_\infty + \sum_{i=1}^m H^i(0)W_t^i), & 0 < t < h \\ -\|u_0\|_\infty + \sum_{i=1}^m H^i(0)W_t^i, & t \geq h. \end{cases}$$

Provided $C > 0$ is large enough, we have $\overline{u}(x, t) = \|u_0\|_\infty + H(0)W_t$ for $t > h/2$ (and similarly for \underline{u}), so these are indeed a super- and sub-solution with the desired properties. In general, the x -dependence and the non-smoothness of u_0 present difficulties.

In what follows, we construct, for $u_0 \in BUC(\mathbb{R}^n)$, a super-solution $\overline{u} = \overline{u}(x, t; u_0)$ of (1) satisfying $\overline{u}^* = u_0$ on $\mathbb{R}^n \times \{0\}$. The construction for the sub-solution is analogous.

6.1. Step 1: Smooth initial conditions. We first construct $\overline{u}(\cdot; \phi)$ for $\phi \in C_b^2(\mathbb{R}^n)$.

First, for some $h > 0$, there exists a smooth-in- x solution $\Phi(x, t) = S(t, 0)\phi(x)$ of (7) defined on $\mathbb{R}^n \times [0, h]$. Set

$$R := \sup_{0 \leq t \leq h} \|\Phi(\cdot, t)\|_{C_b^2(\mathbb{R}^n)} \quad \text{and} \quad C := \sup_{|X|+|p|+|u| \leq R} F(X, p, u, x, t),$$

which are both finite in view of (3) and the continuity of $S(t, 0)$ on $C_b^2(\mathbb{R}^n)$. Taking $\overline{u}(x, t; \phi) := \Phi(x, t) + Ct$ then gives, for $(x, t) \in \mathbb{R}^n \times [0, h]$,

$$F(D^2\Phi(x, t), D\Phi(x, t), \Phi(x, t) + Ct, x, t) \leq F(D^2\Phi(x, t), D\Phi(x, t), \Phi(x, t), x, t) \leq C.$$

This implies that \overline{u} is a super-solution in $\mathbb{R}^n \times [0, h]$.

We now set, for $0 < s \leq t$, $\Phi_0(x, t, s) := S(t, s)(0)(x)$, which, for some $h_0 > 0$ independent of s , is in $C_b^2(\mathbb{R}^n)$ whenever $s \leq t \leq s + h_0$. Since $S(t, s)$ commutes with constants, we have, for all $M \in \mathbb{R}$, $S(t, s)(M)(x) = \Phi_0(x, t, s) + M$.

Now define

$$R_0 := \sup_{0 \leq t-s \leq h_0} \|\Phi_0(\cdot, t, s)\|_{C_b^2(\mathbb{R}^n)}, \quad C_0 := \sup_{|X|+|p|+|u| \leq R_0} F(X, p, u, x, t), \quad \text{and} \quad M_0 := \sup_{x \in \mathbb{R}^n} \bar{u}(x, h; \phi)_+.$$

We construct $\bar{u}(x, t; \phi)$ for $t > h$ as follows. For $k = 0, 1, 2, 3, \dots, \left\lfloor \frac{T-h}{h_0} \right\rfloor$ and $h + kh_0 < t \leq \min(h + (k+1)h_0, T)$, define

$$\bar{u}(x, t; \phi) := \Phi_0(x, t, h + kh_0) + M_k + C_0(t - h - kh_0) \quad \text{and} \quad M_k := \sup_{x \in \mathbb{R}^n} \bar{u}(x, h + kh_0; \phi)_+.$$

The lower-semicontinuity of \bar{u} follows from the definition of M_k . Since, for all k , $M_k \geq 0$ and \bar{u} is smooth on $\mathbb{R}^n \times (h + kh_0, h + (k+1)h_0]$, \bar{u} is a super-solution on all of $\mathbb{R}^n \times [0, T]$.

6.2. Step 2: general initial data. Assume now that $u_0 \in BUC(\mathbb{R}^n)$. Fix $\epsilon > 0$, and choose $\delta = \delta(\epsilon) > 0$ such that $|x - y| < \delta$ implies $|u_0(x) - u_0(y)| < \epsilon$. Set $\psi(x) := 1 - \exp(-|x|^2)$ and $C_\epsilon := 2\|u_0\|_\infty \psi(\delta)^{-1}$.

For $y \in \mathbb{R}^n$, define $U_{y,\epsilon}(x, t) := \bar{u}(x, t; u_0(y) + \epsilon + C_\epsilon \psi(\cdot - y))$ as in the previous subsection. Note that $U_{y,\epsilon}(y, 0) = u_0(y) + \epsilon$, while, for any $x \in \mathbb{R}^n$, $U_{y,\epsilon}(x, 0) = u_0(y) + \epsilon + C_\epsilon \psi(x - y) \geq u_0(x)$ in view of the choice of δ and C_ϵ . It follows that $U(x, t) := \inf_{\epsilon > 0} \inf_{y \in \mathbb{R}^n} U_{y,\epsilon}(x, t)$ satisfies $U(x, 0) = u_0(x)$ for all $x \in \mathbb{R}^n$.

We claim that U_* is the desired super-solution. Let $\eta \in C_b^2(\mathbb{R}^n)$ satisfy $\eta \leq u_0$. Just as in the previous subsection, we can construct a sub-solution $\underline{u}(x, t; \eta)$ with $\underline{u}(x, 0; \eta) = \eta(x)$ and such that \underline{u} is smooth in $\mathbb{R}^n \times [0, h_1]$ for some $h_1 > 0$. Applying the comparison principle to \underline{u} and $U_{y,\epsilon}$ for each y and ϵ yields, for all $(x, t) \in \mathbb{R}^n \times [0, T]$, $\underline{u}(x, t; \eta) \leq U(x, t)$, and since $\inf_{\mathbb{R}^n \times [0, T]} \underline{u}(x, t; \eta) > -\infty$, it follows that $U_* > -\infty$, so that Proposition 1 implies that U_* is a super-solution of (1). Moreover, since \underline{u} is continuous on $\mathbb{R}^n \times [0, h_1]$, $\eta \leq U_*(\cdot, 0) \leq u_0$ on \mathbb{R}^n . Since η was arbitrary, it follows that $U_*(x, 0) = u_0(x)$.

We now show that $(U_*)^* = u_0$ on $\mathbb{R}^n \times \{0\}$. For any $y \in \mathbb{R}^n$ and $\epsilon > 0$,

$$U_*(x, t) \leq U(x, t) \leq U_{y,\epsilon}(x, t).$$

For some $h_2 > 0$, $U_{y,\epsilon}$ is continuous in $\mathbb{R}^n \times [0, h_2]$. Therefore,

$$u_0(y) \leq (U_*)^*(y, 0) \leq U_{y,\epsilon}(y, 0) = u_0(y) + \epsilon.$$

Since ϵ was arbitrary, we have $(U_*)^* = u_0$ on $\mathbb{R}^n \times \{0\}$.

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